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R 110

Deformation of Bearing Surfaces. II

1952

## Deformation of bearing surfaces, II.

### 1. Introduction.

The following computations were performed on request of the Royal Shell Laboratory at Delft. (See your introductory commission of March 15<sup>th</sup>, 1951).

The subject is concerned with the elastic deformation of bearing surfaces, embodied in the following reports:

- (1) The elastic deformation of the surface of a half space under the hydrodynamic pressure distribution, going with a parabolic lubrication-film (IM 524) by H. Blok and J.W. Cohen.
- (2) On the elastic hydrodynamic problem of the lubrication of spur gear teeth. By H. Blok.
- (3) Deformation of bearing surfaces, R 51 by the Computation Department of the Mathematical Centre.

### 2. Method and range of the computations.

The specified task was:

- 1 . To determine  $x_1$  in the neighbourhood of  $x = k$  in such a way that

$$y''(x_1) = 0, \quad (2,1)$$

to state the values

$$y(x_1), y'(x_1) \text{ and } y''(x_1),$$

and the function  $F(x, x_1)$  defined by (see (2) formula (64)):

$$F(x, x_1) = (x - x_1)(x + x_1 - 2 z_\infty) + \frac{2}{y'''(x_1)} \left[ y(x_1) - y(x) \right], \quad (2,2)$$

for the area  $x = -2.0(0.1)7.0$ .

- 2 . To determine the function  $\gamma(G) = \frac{1+k^2}{2G} \int_{-k}^{\infty} \ln \left\{ 1-G(t) \right\}^{-1} dt, \quad (2,3)$

in which

$$\varphi(t) = A \cdot t(1+t^2)^{-2} - B(1+t^2)^{-1} + B\left(\frac{\pi}{2} - \arctan t\right), \quad (2,4)$$

$$A = 2(1+k^2) \quad (2,5)$$

$$B = 1 - 3k^2 \quad (2,6)$$

$$1 + k^2 = 1.225748441 \quad (2,7)$$

and this for a number of further specified values of the parameter  $G$  and at the same time for  $N = 0(0.2)12.2$ , the relation between  $G$  and  $N$  being given by:

$$N = \sqrt{24} \cdot (1 + k^2)^{-\frac{3}{2}} \cdot G \left[ \mathcal{J}(G) \right]^{\frac{3}{2}} \quad (2,8)$$

It was necessary to compute separately the values of  $N$  for

$$G = \left[ \pi (1 - 3 k^2) \right]^{-1}. \quad (\text{For the formulas see (2)}).$$

3 . To determine the function  $y(x, G)$ , defined by

$$y(x, G) = \frac{1}{G} \int_{-k}^{\infty} \ln \left[ 1 - G \varphi(t) \right]^{-1} \ln(x-t)^2 dt, \quad (2,9)$$

for  $x = k$  and  $x = 0$ ,

and  $G$  having the values that correspond with  $N = 2(2)12$ .

Finally the first and second derivative with respect to  $x$  of the function  $y(x, G)$  was computed,  $x$  and  $G$  having the values mentioned above.

For the computations the improper integrals (2,3) and (2,9) were transformed by means of the transformation

$$\operatorname{arccot} t = s. \quad (2,10)$$

$\varphi(t)$  changes into the function  $p(s)$

$$p(s) = -\frac{1+k^2}{4} \sin 4s + 2k^2 \sin 2s + (1-3k^2)s. \quad (2,11)$$

Now:

$$\mathcal{J}(G) = -\frac{1+k^2}{2G} \int_0^{\frac{\pi}{2} + \arctan k} \log \left[ 1 - G p(s) \right] \frac{ds}{\sin^2 s}, \quad (2,12)$$

and

$$y(x, G) = -\frac{1}{G} \int_0^{\frac{\pi}{2} + \arctan k} \log \left[ 1 - G p(s) \right] \log \left| x - \cot s \right|^2 \frac{ds}{\sin^2 s} \quad (2,13)$$

The integral  $\mathcal{J}(G)$  is not improper in the point  $s = 0$ , for

$$\frac{\log \left\{ 1 - G p(s) \right\}^{-1}}{\sin^2 s} = \frac{8}{3} G s \left\{ 1 - \frac{7+9k^2}{15} s^2 \dots \right\} \quad (2,14)$$

For  $s_0 = \operatorname{arccot} k$ , the integrand of (2,12) contains a logarithmic singularity, amounting to

$$-\frac{2 \log(s-s_0)}{\sin^2 s_0}.$$

This difficulty has been surmounted by the method described in (3).

The integrand in (2,13) also contains a logarithmic singularity, in  $\cot s_0 = x$ , in fact.

This logarithmic character of the singularity disappears in evaluating the derivatives with respect to  $x$ .

It is easily derived that:

$$y'(x, G) = - \frac{2}{G} \int_0^{\frac{\pi}{2} + \arctan k} \frac{\log[1 - G p(s)]}{(x - \cot s) \sin^2 s} ds \quad (2,15)$$

which is a principal value in the sense of Cauchy.

Now we expand the integrand into a series

$$\frac{a_{-1}}{u} + a_0 + a_1 u + a_2 u^2 \dots, \quad (2,16)$$

in which  $u = s - s_0$ , and

$$s_0 = \operatorname{arccot} x \quad (2,17)$$

and now we evaluate  $a_{-1} \int_0^{\frac{\pi}{2} + \arctan k} \frac{1}{s} ds$  analytically and numerically. The difference between the two evaluations is the error that is made by determining (2,15) via the way of numerical integration.

The formula for  $y''(x, G)$  is:

$$y''(x, G) = \frac{2}{G} \int_0^{\frac{\pi}{2} + \arctan k} \left\{ \log[1 - G p(s)] - \log[1 - G p(s_0)] \right\} \frac{ds}{(x - \cot s)^2 \sin^2 s}. \quad (2,18)$$

(2,18) can be proved by writing the integral (2,15) as follows:

$$\begin{aligned} y'(x, G) = - \frac{2}{G} \left\{ \int_0^{s_0 - \xi} \frac{\log[1 - G p(s)]}{(x - \cot s) \sin^2 s} ds + \right. \\ \left. + \int_{s_0 - \xi}^{s_0 + \xi} \frac{\log[1 - G p(s)]}{(x - \cot s) \sin^2 s} ds + \right. \\ \left. + \int_{s_0 + \xi}^{\frac{\pi}{2} + \arctan k} \frac{\log[1 - G p(s)]}{(x - \cot s) \sin^2 s} ds \right\}, \end{aligned}$$

and then performing the differentiation with respect to  $x$ .

$$\begin{aligned} \frac{d}{dx} \left[ \int_{s_0 - \xi}^{s_0 + \xi} \frac{\log[1 - G p(s)]}{(x - \cot s) \sin^2 s} ds \right] &= \frac{d}{dx} \left[ \int_{-\xi}^{\xi} \left( \frac{a_{-1}}{u} + a_0 + a_1 u + \dots \right) du \right] \\ &= \frac{d}{dx} \left[ 2 a_0 \xi + \frac{2}{3} a_2 \xi^3 + \dots \right] = 2 b_0 \xi + \frac{2}{3} \xi^3 + \dots \end{aligned}$$

in which  $b_n = \frac{da_n}{dx}$ .

Therefore,

$$\lim_{\xi \rightarrow \infty} \frac{d}{dx} \left[ \int_{s_0-\xi}^{s_0+\xi} \frac{\log [1-G p(s)]}{x \cot s \sin^2 s} ds \right] = 0.$$

Utilizing

$$x - \cot s_0 = 0$$

we find  $\frac{ds_0}{dx} = -\sin^2 s_0$  and therefore differentiation of the first integral yields:

$$\begin{aligned} & + \frac{2}{G} \int_0^{s_0-\xi} \frac{\log [1-G p(s)]}{(x - \cot s)^2 \sin^2 s} ds + \frac{2}{G} \frac{\log [1-G p(s_0-\xi)]}{[x - \cot (s_0-\xi)] \sin^2 (s_0-\xi)} \sin^2 s_0 \\ & = + \frac{2}{G} \int_0^{s_0-\xi} \left\{ \log [1-G p(s)] - \log [1-G p(s_0-\xi)] \frac{\sin^2 s_0}{\sin^2 (s_0-\xi)} \right\} \frac{ds}{(x - \cot s)^2 \sin^2 s} \end{aligned}$$

When, in a similar manner, we differentiate the last integral and assume that  $\lim \xi \rightarrow 0$  we find (2,18).

For the case  $x = 0$ , this new-found integral is again an integral of which the principal value has to be considered, for  $x = k$  the integrand is regular.

### 3. Numerical results.

All results have an error of at most one unit of the last figure that is stated. Here and there it proved necessary to use more figures than was necessary for the final answer, in view of the fact that figures sometimes drop off.

For  $x_1$  was found

$$x_1 = 0.384043$$

and

$$y(x_1) = -2.5714$$

$$y'(x_1) = -2.18004$$

$$y''(x_1) = 10.53252.$$

In the tables I, II and III the functions  $F(x, x_1)$ ,  $f(G)$  as functions of  $G$  and  $N$ , and  $y(x, G)$ ,  $y'(x, G)$  and  $y''(x, G)$  follow underneath.

| $1 \leq$ | $F(x, x_1)$ |     | $F(x, x_1)$ |     | $F(x, x_1)$ |
|----------|-------------|-----|-------------|-----|-------------|
| $x$      |             | $x$ |             | $x$ |             |
| - 2,0    | 5,49315     | 2,1 | 1,68235     | 6,2 | 29,95236    |
| - 1,9    | 5,00686     | 2,2 | 1,95412     | 6,3 | 31,07174    |
| - 1,8    | 4,54144     | 2,3 | 2,24735     | 6,4 | 32,21136    |
| - 1,7    | 4,09699     | 2,4 | 2,56202     | 6,5 | 33,37119    |
| - 1,6    | 3,67358     | 2,5 | 2,89806     | 6,6 | 34,55125    |
| - 1,5    | 3,27132     | 2,6 | 3,25545     | 6,7 | 35,75152    |
| - 1,4    | 2,89032     | 2,7 | 3,63410     | 6,8 | 36,97198    |
| - 1,3    | 2,53075     | 2,8 | 4,03398     | 6,9 | 38,21265    |
| - 1,2    | 2,19276     | 2,9 | 4,45501     | 7,0 | 39,47349    |
| - 1,1    | 1,87655     | 3,0 | 4,89713     |     |             |
| - 1,0    | 1,58239     | 3,1 | 5,36031     |     |             |
| - 0,9    | 1,31063     | 3,2 | 5,84446     |     |             |
| - 0,8    | 1,06163     | 3,3 | 6,34955     |     |             |
| - 0,7    | 0,83600     | 3,4 | 6,87554     |     |             |
| - 0,6    | 0,63453     | 3,5 | 7,42235     |     |             |
| - 0,5    | 0,45845     | 3,6 | 7,98997     |     |             |
| - 0,4    | 0,31059     | 3,7 | 8,57835     |     |             |
| - 0,3    | 0,19416     | 3,8 | 9,18745     |     |             |
| - 0,2    | 0,10949     | 3,9 | 9,81722     |     |             |
| - 0,1    | 0,05378     | 4,0 | 10,46764    |     |             |
| 0        | 0,02173     | 4,1 | 11,13868    |     |             |
| 0,1      | 0,00648     | 4,2 | 11,83031    |     |             |
| 0,2      | 0,00110     | 4,3 | 12,54249    |     |             |
| 0,3      | 0,00004     | 4,4 | 13,27520    |     |             |
| 0,4      | 0,00001     | 4,5 | 14,02842    |     |             |
| 0,5      | 0,00014     | 4,6 | 14,80212    |     |             |
| 0,6      | 0,00164     | 4,7 | 15,59627    |     |             |
| 0,7      | 0,00694     | 4,8 | 16,41088    |     |             |
| 0,8      | 0,01902     | 4,9 | 17,24589    |     |             |
| 0,9      | 0,04091     | 5,0 | 18,10132    |     |             |
| 1,0      | 0,07533     | 5,1 | 18,97713    |     |             |
| 1,1      | 0,12459     | 5,2 | 19,87329    |     |             |
| 1,2      | 0,19057     | 5,3 | 20,78982    |     |             |
| 1,3      | 0,27466     | 5,4 | 21,72669    |     |             |
| 1,4      | 0,37798     | 5,5 | 22,68390    |     |             |
| 1,5      | 0,50130     | 5,6 | 23,66140    |     |             |
| 1,6      | 0,64515     | 5,7 | 24,65920    |     |             |
| 1,7      | 0,80994     | 5,8 | 25,67731    |     |             |
| 1,8      | 0,99594     | 5,9 | 26,71568    |     |             |
| 1,9      | 1,20330     | 6,0 | 27,77431    |     |             |
| 2,0      | 1,43207     | 6,1 | 28,85321    |     |             |

| $2 \underline{a}$<br>G | $\gamma(G)$ | $2 \underline{b}$<br>N | $\gamma(G)$ | $2 \underline{b}$<br>N | $\gamma(G)$ |
|------------------------|-------------|------------------------|-------------|------------------------|-------------|
| 0,0                    | 1           | 0,0                    | 1           | 6,4                    | 1,61162     |
| 0,1                    | 1,03263     | 0,2                    | 1,01717     | 6,6                    | 1,63275     |
| 0,2                    | 1,06935     | 0,4                    | 1,03447     | 6,8                    | 1,65400     |
| 0,3                    | 1,11112     | 0,6                    | 1,05192     | 7,0                    | 1,67538     |
| 0,4                    | 1,15944     | 0,8                    | 1,06948     | 7,2                    | 1,69687     |
| 0,5                    | 1,21655     | 1,0                    | 1,08717     | 7,4                    | 1,71848     |
| 0,6                    | 1,28609     | 1,2                    | 1,10498     | 7,6                    | 1,74021     |
| 0,69                   | 1,36458     | 1,4                    | 1,12292     | 7,8                    | 1,76206     |
| 0,7                    | 1,37456     | 1,6                    | 1,14100     | 8,0                    | 1,78404     |
| 0,8                    | 1,49587     | 1,8                    | 1,15919     | 8,2                    | 1,80613     |
| 0,84                   | 1,56041     | 2,0                    | 1,17750     | 8,4                    | 1,82834     |
| 0,9                    | 1,69062     | 2,2                    | 1,19593     | 8,6                    | 1,85067     |
| 0,97                   | 1,99381     | 2,4                    | 1,21449     | 8,8                    | 1,87312     |
| 0,98                   | 2,10312     | 2,6                    | 1,23319     | 9,0                    | 1,89569     |
| 0,98622858             | 2,31253     | 2,8                    | 1,25200     | 9,2                    | 1,91837     |
|                        |             | 3,0                    | 1,27094     | 9,4                    | 1,94118     |
|                        |             | 3,2                    | 1,29000     | 9,6                    | 1,96411     |
|                        |             | 3,4                    | 1,30918     | 9,8                    | 1,98716     |
|                        |             | 3,6                    | 1,32848     | 10,0                   | 2,01033     |
|                        |             | 3,8                    | 1,34791     | 10,2                   | 2,03362     |
|                        |             | 4,0                    | 1,36746     | 10,4                   | 2,05703     |
|                        |             | 4,2                    | 1,38713     | 10,6                   | 2,08056     |
|                        |             | 4,4                    | 1,40693     | 10,8                   | 2,10420     |
|                        |             | 4,6                    | 1,42686     | 11,0                   | 2,12797     |
|                        |             | 4,8                    | 1,44690     | 11,2                   | 2,15186     |
|                        |             | 5,0                    | 1,46706     | 11,4                   | 2,17587     |
|                        |             | 5,2                    | 1,48735     | 11,6                   | 2,20000     |
|                        |             | 5,4                    | 1,50776     | 11,8                   | 2,22427     |
|                        |             | 5,6                    | 1,52829     | 12,0                   | 2,24861     |
|                        |             | 5,8                    | 1,54894     | 12,2                   | 2,27310     |
|                        |             | 6,0                    | 1,56972     |                        |             |
|                        |             | 6,2                    | 1,59061     |                        |             |

|            |          |
|------------|----------|
| G          | N        |
| 0,98622858 | 12,52021 |

3 a

| N  | G       | $y(x, G)$ | $y'(x, G)$ | $y''(x, G)$       |
|----|---------|-----------|------------|-------------------|
| 2  | 0.43360 | - 3.6565  | - 1.4716   | 12.32             |
| 4  | 0.69292 | - 4.7485  | - 1.7283   | 19.37             |
| 6  | 0.84514 | - 6.052   | - 2.0035   | 31.78             |
| 8  | 0.93001 | - 7.623   | - 2.2970   | 57.01             |
| 10 | 0.97191 | - 9.584   | - 2.6097   | 127.8             |
| 12 | 0.98580 | -12.27    | - 2.974    | $8.4 \times 10^2$ |

3 b

|    | G       | $y(x, G)$ | $y'(x, G)$ | $y''(x, G)$ |
|----|---------|-----------|------------|-------------|
| 2  | 0.43360 | - 1.4594  | - 6.3674   | - 2.471     |
| 4  | 0.69292 | - 1.8887  | - 7.8194   | - 5.364     |
| 6  | 0.84514 | - 2.3578  | - 9.3633   | - 8.920     |
| 8  | 0.93001 | - 2.8613  | -10.9726   | -12.81      |
| 10 | 0.97191 | - 3.403   | -12.628    | -16.76      |
| 12 | 0.98580 | - 3.972   | -14.33     | -20.7       |